Your Name

Your Signature

Student ID #

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Graphing calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. Make sure that it is easy for graders to follow what you are doing.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- **You may write on the backs of pages** (and are expected to for some questions, in order to have enough space). Both sides of each page of the exam will be scanned.
- Raise your hand if you have a question.
- This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	11	
2	11	
3	12	
4	16	
Total	50	

1. (11 points) **For this problem only, you do not have to show work or justification.** For each of the following statements, circle "T" to the left if the statement is true, and "F" if the statement is false. **Here "true" means "always true".** If the are both examples of and counterexamples to the statement, the correct answer is "false." If you don't know the answer almost immediately, just make a guess and move on; time is better spent on the other exam questions.

$\boxed{T}$	F	If $T(\mathbf{x}) = A\mathbf{x}$ is both injective and surjective, then A is invertible.
$\boxed{T}$	F	If A is not square then $T(\mathbf{x}) = A\mathbf{x}$ is not invertible.
T	$\boxed{F}$	If A is a 3×3 matrix and $A^2 = 0_{3\times3}$, then $A = 0_{3\times3}$ (here $0_{3\times3}$ is the 3×3 matrix filled with all zeros.)
T	$\boxed{F}$	If $T : \mathbb{R}^m \to \mathbb{R}^n$ is surjective then $n > m$.
$\boxed{T}$	F	If $S \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and there is some $\mathbf{v} \in \mathbb{R}^n$ which is *not* in S, then $\dim S \leq n-1$.
T	$\boxed{F}$	If $S \subset \mathbb{R}^3$ is a subspace and $\mathbf{v}$ and $\mathbf{w}$ are two linearly independent vectors which are *not* in S, then $\dim S \leq 3-2 = 1$.
T	$\boxed{F}$	If A and B are 4×4 matrices and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^4$, then $(A+B)(\mathbf{v}+\mathbf{w}) = A\mathbf{v} + B\mathbf{w}$.
$\boxed{T}$	F	If $S \subset S' \subset \mathbb{R}^m$ are subspaces, then $\dim S \leq \dim S' \leq m$.
T	$\boxed{F}$	Suppose that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ spans a subspace S. Then $\dim S = 4$.
$\boxed{T}$	F	Suppose that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is a basis for S. Then every sublist of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is independent.
T	$\boxed{F}$	If $T : \mathbb{R}^m \to \mathbb{R}^n$ is not injective, then it is surjective.

2. (11 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

(a) Compute A^{-1}.

(b) Suppose

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Solve the equation $AXA = A + BA$ for the matrix X. Give your final answer as an explicit matrix.

(c) Suppose that we know that

$$A\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\mathbf{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A\mathbf{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the vectors $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$. (Note: Your computation for this part should mostly be done already.)

Solution.

(a) The augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

row reduces to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right],$$

so

$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}.$$

[Your solution should include the row operations you performed to compute this.]

(b) Multiplying both sides of the given equation first on the right and then on the left by A^{-1}, we have:

$$\begin{aligned} AX &= (AXA)A^{-1} = (A+BA)A^{-1} = AA^{-1} + BAA^{-1} = I_3 + B \\ X &= A^{-1}(AX) = A^{-1}(I_3 + B). \end{aligned}$$

We compute

$$B + I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so

$$X = A^{-1}(B + I_3) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 0 & 0 \\ -6 & 0 & 0 \end{bmatrix}.$$

(c) Multiplying both sides of each equation on the left by A^{-1} gives

$$\mathbf{x_1} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x_2} = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x_3} = A^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and multiplying a matrix by the ith standard unit vector just gives the ith column of the matrix, so we see that

$$\mathbf{x_1} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{x_2} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

3. (12 points) Consider the subspace

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 : \begin{matrix} x_1 + 2x_3 - x_4 + x_5 = 0 \\ x_2 - x_3 + x_4 = 0 \end{matrix} \right\}.$$

(a) Give a basis for S. What is $\dim S$?
Solution. The system is already in reduced echelon form, and writing the pivot variables in terms of the free variables (as we usually do to write a solution set in parametric form) shows us that the set of solutions to the system is

$$\left\{ t_1 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Therefore the list

$$\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is a basis for S (since it spans S and is linearly independent). We then see $\dim S = 3$.

(b) Find a list of vectors which spans S but is *not* a basis.
Solution. Adding any vector in S to our basis above will produce a spanning set for S which is not linearly independent (and therefore not a basis). For example, we can choose

$$\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Find a matrix A with five rows such that $S = \operatorname{row}(A)$.
Solution. We just need to produce any matrix with 5 rows whose rows form a spanning list for S (for example, any list of 5 vectors in S which contains the list we found in (a)). For example, we can take

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(d) Find an linear transformation T which is *not* injective (one-to-one) such that $S = \operatorname{range}(T)$. Is T surjective (onto)?
Solution. We can specify a linear transformation by specifying its matrix. Since the range of a transformation is equal to the column space of its matrix, to have $\operatorname{range}(T) = S$, we just need to

choose a matrix with columns spanning S. To have that T is not injective, we just need that the columns of its matrix are not linearly independent (so they do not form a basis for S). So, using our example from (b), for instance, we can take $T : \mathbb{R}^4 \to \mathbb{R}^5$ defined by

$$T(\mathbf{x}) = \begin{bmatrix} -2 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

Since $\operatorname{range}(T) = S \neq \mathbb{R}^5$, T is not surjective.

(e) Find a matrix B such that $S = \operatorname{null}(B)$. What is the rank of B?
Solution. We can find B immediately by just rewriting the system of equations defining S as a matrix equation of the form $B\mathbf{x} = \mathbf{0}$; so $S = \operatorname{null}(B)$ where

$$B = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

Since $\operatorname{nullity}(B) = \dim S = 3$, the Rank-Nullity Theorem tells us that $\operatorname{rank}(B) = 5 - 3 = 2$.

4. (16 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Consider the sub**set**

$$C = \{\mathbf{v} \in \mathbb{R}^2 : \|T(\mathbf{v})\| \leq \|\mathbf{v}\|\},$$

where here $\|\mathbf{v}\|$ is the length of the vector $\mathbf{v}$; i.e. $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{x^2+y^2}$. That is, C is the set of all vectors which are made shorter (or kept the same length) when input to T.*

(a) Write down the three conditions a subset $S \subset \mathbb{R}^2$ has to satisfy in order to be a sub**space**.

(b) Show that C satisfies *two* of the three conditions you wrote down in part (a). (You do not have to show whether or not it satisfies the remaining condition). Hint: the geometric interpretation of vector operations should help you here.

(c) Suppose (for this part and all following parts of the problem) that

$$T(\mathbf{x}) = \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}.$$

Note that then $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in C$ and $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin C$. Find another vector $\mathbf{w} \in C$ which is linearly independent from $\mathbf{e_1}$.

(d) Suppose that S is a subspace which contains C. What is the smallest possible dimension that S could have?† Explain. Give an example of such an S with this dimension.

(e) Suppose that S' is a subspace which is contained inside C. What is the largest possible dimension that S' could have? Explain. Give an example of such an S' with this dimension.

Solution.

(a) The 3 properties a subset S must satisfy in order to be a subspace are:

$\mathbf{0} \in S$.

If $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{u}+\mathbf{v} \in S$ (closure under vector addition).

If $\mathbf{v} \in S$ and $c \in \mathbb{R}$, then $c\mathbf{v} \in S$ (closure under scalar multiplication).

(b) Since T is linear, $T(\mathbf{0}) = \mathbf{0}$, and so $\|T(\mathbf{0})\| \leq \|\mathbf{0}\|$ $(0 \leq 0)$. Therefore, $\mathbf{0} \in C$.
Next, suppose that $\mathbf{v} \in C$, that is, $\|T(\mathbf{v})\| \leq \|\mathbf{v}\|$, and $c \in \mathbb{R}$. Since multiplying a vector $\mathbf{v}$ by a scalar c scales the length of a vector by a factor of $|c|$, and because linear transformations respect scalar multiplication, we see that

$$\|T(c\mathbf{v})\| = \|cT(\mathbf{v})\| = |c|\|T(\mathbf{v})\| \leq |c|\|\mathbf{v}\| = \|c\mathbf{v}\|.$$

(For the inequality, we used the fact that $\|T(\mathbf{v})\| \leq \|\mathbf{v}\|$ by assumption). This shows that $c\mathbf{v} \in C$. Overall, we have shown that C is closed under scalar multiplication.

(c) Vectors along the $\mathbf{e_1}$ direction are shrunk, and vectors along the $\mathbf{e_2}$ direction are expanded, so we should look for a vector which is independent from $\mathbf{e_1}$ but is almost pointed in the $\mathbf{e_1}$ direction. For instance, if we take

$$\mathbf{w} = \begin{bmatrix} 100 \\ 1 \end{bmatrix}$$

then

$$\|T(\mathbf{w})\| = \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{5} \leq \sqrt{100^2+1} = \left\| \begin{bmatrix} 100 \\ 1 \end{bmatrix} \right\| = \|\mathbf{w}\|.$$

($\mathbf{w}$ is also clearly linearly independent from $\mathbf{e_1}$ since neither is a multiple of the other.)

*You don't have to compute any square roots to do this problem. It might help to remember that whenever $a \leq b$, we know $\sqrt{a} \leq \sqrt{b}$.

†It might help you to draw a picture of what you think C looks like in order to do these last two parts.

(d) If a subspace S contains C, S is a subspace of $\mathbb{R}^2$ which contains the independent list of vectors $\mathbf{e_1}, \mathbf{w}$. Since it contains an independent list of size 2, it must have dimension at least 2. An example of a subspace S containing C (in fact the only example) is $S = \mathbb{R}^2$.

(e) If S' is contained inside C, S' is a subspace of $\mathbb{R}^2$, so its dimension must be $0, 1$, or 2. If S had dimension 2, then it would be all of $\mathbb{R}^2$; but then S' would contain $\mathbf{e_2}$, which does not lie inside C, so this is impossible. Therefore, S' has dimension at most 1. An example of a subspace S' of dimension 1 contained inside of C is $S' = \text{span}(\mathbf{e_1})$.